Sure! Here’s an enhanced list with examples and problems for each case. I'll provide examples to illustrate the concept, and I'll also suggest some problems you can try from popular problem-solving platforms.

### 1. \*\*Divisibility Rules\*\*

\*\*Example:\*\*

- \*\*Divisibility by 3:\*\* To check if 123 is divisible by 3, sum the digits: \(1 + 2 + 3 = 6\). Since 6 is divisible by 3, so is 123.

- \*\*Divisibility by 7:\*\* To check if 343 is divisible by 7, double the last digit and subtract it from the rest of the number: \(34 - 6 = 28\). Since 28 is divisible by 7, so is 343.

\*\*Problems:\*\*

- [Project Euler Problem 1](https://projecteuler.net/problem=1): Multiples of 3 and 5.

- [Codeforces Problem 405A](https://codeforces.com/problemset/problem/405/A): Gravity Flip.

### 2. \*\*Greatest Common Divisor (GCD) and Least Common Multiple (LCM)\*\*

\*\*Example:\*\*

- \*\*GCD:\*\* Find the GCD of 48 and 18 using the Euclidean algorithm:

\[

\text{GCD}(48, 18) = \text{GCD}(18, 48 \mod 18) = \text{GCD}(18, 12) = \text{GCD}(12, 6) = \text{GCD}(6, 0) = 6

\]

- \*\*LCM:\*\* Calculate the LCM of 12 and 15 using the relation \( \text{LCM}(a, b) = \frac{|a \times b|}{\text{GCD}(a, b)} \):

\[

\text{LCM}(12, 15) = \frac{12 \times 15}{\text{GCD}(12, 15)} = \frac{180}{3} = 60

\]

\*\*Problems:\*\*

- [Codeforces Problem 822A](https://codeforces.com/problemset/problem/822/A): I'm bored with life.

- [HackerRank Problem](https://www.hackerrank.com/challenges/lcm-challenge/problem): LCM Challenge.

### 3. \*\*Modular Arithmetic\*\*

\*\*Example:\*\*

- \*\*Modulus Operation:\*\* Calculate \( 7^4 \mod 5 \):

\[

7^4 = 2401 \quad \text{and} \quad 2401 \mod 5 = 1

\]

- \*\*Modular Inverse:\*\* Find the modular inverse of 3 modulo 7:

\[

3x \equiv 1 \ (\text{mod} \ 7)

\]

Testing values, we find that \( x = 5 \) works since \( 3 \times 5 = 15 \equiv 1 \ (\text{mod} \ 7) \).

\*\*Problems:\*\*

- [Codeforces Problem 1352C](https://codeforces.com/problemset/problem/1352/C): K-th Not Divisible by n.

- [LeetCode Problem 371](https://leetcode.com/problems/sum-of-two-integers/): Sum of Two Integers (involves modular arithmetic concepts).

### 4. \*\*Fermat's Little Theorem\*\*

\*\*Example:\*\*

- Calculate \( 3^{100} \mod 7 \) using Fermat’s Little Theorem. Since 7 is prime:

\[

3^{6} \equiv 1 \ (\text{mod} \ 7) \quad \text{so} \quad 3^{100} \equiv 3^{100 \mod 6} \equiv 3^4 \equiv 81 \equiv 4 \ (\text{mod} \ 7)

\]

\*\*Problems:\*\*

- [HackerRank Problem](https://www.hackerrank.com/challenges/fermats-little-theorem/problem): Fermat's Little Theorem.

- [Codeforces Problem 1006E](https://codeforces.com/problemset/problem/1006/E): Military Problem.

### 5. \*\*Chinese Remainder Theorem\*\*

\*\*Example:\*\*

- Solve the system:

\[

x \equiv 2 \ (\text{mod} \ 3)

\]

\[

x \equiv 3 \ (\text{mod} \ 5)

\]

\[

x \equiv 2 \ (\text{mod} \ 7)

\]

Start with \( x = 2 \) mod 3. Test \( x = 23 \), as it satisfies all three conditions.

\*\*Problems:\*\*

- [Codeforces Problem 476B](https://codeforces.com/problemset/problem/476/B): Dreamoon and WiFi.

- [Project Euler Problem 531](https://projecteuler.net/problem=531): Chinese Remainder Theorem.

### 6. \*\*Euler’s Totient Function\*\*

\*\*Example:\*\*

- Calculate \( \phi(9) \). Since 9 = \(3^2\):

\[

\phi(9) = 9 \times \left(1 - \frac{1}{3}\right) = 6

\]

Six numbers are coprime to 9: 1, 2, 4, 5, 7, 8.

\*\*Problems:\*\*

- [Project Euler Problem 72](https://projecteuler.net/problem=72): Counting fractions.

- [HackerRank Problem](https://www.hackerrank.com/challenges/eulers-totient-function/problem): Euler's Totient Function.

### 7. \*\*Diophantine Equations\*\*

\*\*Example:\*\*

- Solve \( 3x + 5y = 14 \) for integers \( x \) and \( y \).

- The equation is solvable since \( \text{GCD}(3, 5) = 1 \).

- One solution is \( x = 3 \), \( y = 1 \).

\*\*Problems:\*\*

- [Codeforces Problem 289B](https://codeforces.com/problemset/problem/289/B): Polo the Penguin and Matrix.

- [LeetCode Problem 365](https://leetcode.com/problems/water-and-jug-problem/): Water and Jug Problem.

### 8. \*\*Quadratic Residues and Non-Residues\*\*

\*\*Example:\*\*

- \*\*Quadratic Residue:\*\* Check if 3 is a quadratic residue modulo 7:

- Test values: \(1^2 = 1\), \(2^2 = 4\), \(3^2 = 9 \equiv 2 \ (\text{mod} \ 7)\). No perfect square equals 3 modulo 7.

\*\*Problems:\*\*

- [Codeforces Problem 1620D](https://codeforces.com/problemset/problem/1620/D): Triangle Coloring.

- [Project Euler Problem 216](https://projecteuler.net/problem=216): Investigating the primality of numbers of the form \(n^2 + n + 1\).

### 9. \*\*Pigeonhole Principle\*\*

\*\*Example:\*\*

- If you have 10 pairs of socks and you pick 11 socks, at least one pair will have matching socks. This is a direct application of the Pigeonhole Principle.

\*\*Problems:\*\*

- [Codeforces Problem 1004A](https://codeforces.com/problemset/problem/1004/A): Sonya and Hotels.

- [Project Euler Problem 78](https://projecteuler.net/problem=78): Coin partitions.

### 10. \*\*Prime Factorization\*\*

\*\*Example:\*\*

- \*\*Factorization:\*\* Find the prime factorization of 60:

- \( 60 = 2^2 \times 3 \times 5 \).

\*\*Problems:\*\*

- [Codeforces Problem 546B](https://codeforces.com/problemset/problem/546/B): Soldier and Badges.

- [Project Euler Problem 3](https://projecteuler.net/problem=3): Largest prime factor.

### 11. \*\*Sieve of Eratosthenes\*\*

\*\*Example:\*\*

- To find all primes less than 30:

- Start with the smallest prime, 2. Eliminate all multiples of 2.

- Move to the next unmarked number, 3. Eliminate all multiples of 3.

- Continue until you've processed all numbers up to \( \sqrt{30} \).

\*\*Problems:\*\*

- [LeetCode Problem 204](https://leetcode.com/problems/count-primes/): Count Primes.

- [Project Euler Problem 10](https://projecteuler.net/problem=10): Summation of primes.

### 12. \*\*Binary Exponentiation\*\*

\*\*Example:\*\*

- Compute \( 3^{13} \mod 7 \) using binary exponentiation:

- Convert 13 to binary: 1101.

- Compute \( 3^1 \mod 7 = 3 \), \( 3^2 \mod 7 = 9 \equiv 2 \), \( 3^4 \mod 7 = 4 \), \( 3^8 \mod 7 = 16 \equiv 2 \).

- Multiply relevant powers to get \( 3^{13} \equiv 3 \times 4 \times 2 = 24 \equiv 3 \ (\text{mod} \ 7) \).

\*\*Problems